Fundamental Properties of Splitting Numbers of Integer Tiles

Seungmok Lee
Advisors: Gregory Clark & Dr. Joshua Cooper, Department of Mathematics
Collaborators: J. Folks, S. Miyasaki, C. Simmons, D. Stewart.

Introduction

János Pach posed a famous question concerning the splitting number of disks in the plane. In short, the question asked how many times each point in the plane needed to be covered by a unit disk to ensure that there exists a two-coloring of the disks such that each point is beneath a disk of both colors. It was originally proven (although unwritten) by Pach, but was later disproved! This question lead to research into similar problems for convex shapes. However, these results do not extend naturally to the 1-dimensional case of the line.

We explore results similar to the disk-covering problem by discretizing the real line into integers. Instead of considering unit circles we consider integer tiles. We provide upper and lower bounds on the splitting number of a finite tile, and discuss the relationship between the splitting number of a tile and the splitting number of a dilation/translation of that tile. Our main result is a characterization of splitting numbers for tiles of size at most three.

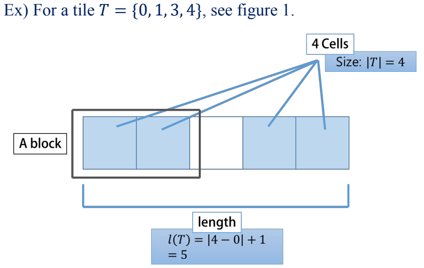
Theoretical Background

1. Tiles

- A **Tile** is a subset of the integers.
- Each element of a tile is called a **cell**.
- Maximal consecutive cells are called **blocks**.
- **The size of T** is the number of elements in T.
- **The length of T** is denoted by $l(T)$, and it is

$$l(T) = |max(T) - min(T)| + 1$$

- Ex) For a tile $T = \{0, 1, 3, 4\}$, see figure 1.



<Figure 1>Anatomy of a tile, cells, and blocks.

2. Coverings

- A **shift** is an integer (denoted by s).
- A **translation** is a tile shifted by a shift s (denoted by $T + \{s\}$).
- A multiset S is a **set of shifts**.
- The **translate set** T_S is a collection of translations.

$$T_S := \{T + \{s\} : s \in S\}$$

- A **set of tiles** *which contain* x is

$$T_S(x) := \{T \in T_S : x \in T\}$$

- T_S **covers** $\mathbb{Z}$ if the union of every tile T in T_S is $\mathbb{Z}$.

$$\bigcup_{T \in T_S} T = \mathbb{Z}$$

- T_S is *a k-covering* if

$$|T_S(x)| \geq k \text{ for all } x \in \mathbb{Z}$$

- T_S is an *exact k-covering* if

$$|T_S(x)| = k \text{ for all } x \in \mathbb{Z}$$

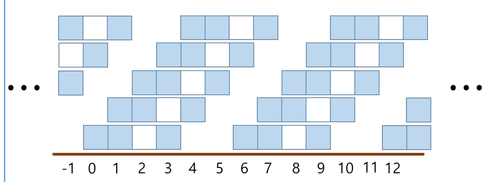
- A covering T_S is **splittable** if S can be partitioned into two multisets A and B such that T_A and T_B are coverings:
  1) T_S, T_A, T_B are coverings
  2) $S \supseteq A \cup B$
  3) $\chi_S(x) = \chi_A(x) + \chi_B(x)$ for all $x \in \mathbb{Z}$
- The **splitting number** of a tile is the least integer k such that every k-covering is splittable (denoted by $\sigma(k)$).

3. Computing Splitting Numbers

- **Theorem 1. Finite tiles have a splitting number of at least 2.**
- $\boldsymbol{\sigma(T) \geq k}$ if there exists an unsplittable $k - 1$ covering.
- $\boldsymbol{\sigma(T) \leq k}$ if every k-coverings is splittable.
- Ex) Show $\sigma(\{0, 1, 3\}) \geq 3$.

Let $S := [0]_6 \cup [1]_6 \cup [2]_6 \cup [3]_6 \cup [4]_6$.

See figure 2. It is an unsplittable 2-covering.



<Figure 2>Proof on unsplittability of T_S. There are 5 translations which are completely in [0, 7]. By the pigeonhole principle, the set A must have at least 3 of those tiles. This directly implies that B must have at most 2 of those tiles. Hence T_B does not cover [0,7]. So T_S is not splittable.

Results

1. General Properties of Splitting Numbers

(1) Stricter upper bound

- **Theorem 2. For every tile $T (|T| > 1)$,**

$$\boldsymbol{\sigma(T) \leq |T|}$$

Ex) $\sigma(\{0, 2, 3, 7\}) \leq 4$, $\sigma(\{0, 4, 5, 6, 7, 9\}) \leq 6$

- **Corollary 1. For every 3-element tile $T = \{0, k, n\}$,**

$$\boldsymbol{2 \leq \sigma(T) \leq 3}$$

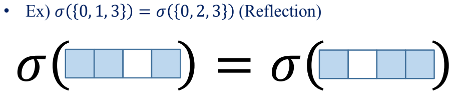
Corollary 1 follows from Theorem 1 and 2.

(2) Splitting number of dilated/translated tiles

- **Theorem 3. For every tile T and for all $0 \neq k \in \mathbb{Z}$,**

$$\boldsymbol{\sigma(T) = \sigma(kT)}$$

- Ex) $\sigma(\{0, 1, 3\}) = \sigma(\{0, 2, 3\})$ (Reflection)



<Figure 3>

- **Theorem 4. For every tile T and for all $k \in \mathbb{Z}$,**

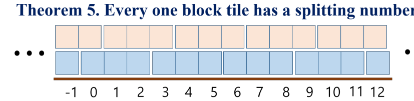
$$\boldsymbol{\sigma(T) = \sigma(T + \{k\})}$$

By Theorem 4, we may assume all tiles begin with 0. Further, by reflection we may assume all tiles have non-negative elements.

2. Splitting Numbers of Elementary Tiles

(1) Splitting number of 1-block tile

- **Theorem 5. Every one block tile has a splitting number of 2.**



<Figure 4> One example of splittable 2-covering of $T = \{0, 1, 2\}$

(2) Splitting number of an arithmetic progression

- **Theorem 6. Every tile which is an arithmetic progression has a splitting number of 2.**

Theorem 6 follows from Theorem 3 and Theorem 5 since every arithmetic progression is a dilation of a 1-block tile.

(3) Splitting number of 2-element tiles

- **Theorem 7. Every 2-element tile has a splitting number of 2.**

Theorem 7 follows from Theorem 3 since every 2-element tile is a dilation of $T_0 = \{0, 1\}$, where $\sigma(T_0) = 2$ (c.f. Theorem 5).

3. Splitting Numbers of 3-element Tiles

- **Theorem 8. A 3-element tile has a splitting number of 2 if and only if the tile is an arithmetic progression.**

Proof: We already know that every arithmetic progression has a splitting number of 2 (c.f. Theorem 6). It remains to show that every 3-element tile which is not an arithmetic progression has a splitting number of 3, in other words, there exists an unsplittable 2-covering of such a tile. We can write every 3-element tile T as follows:

$$T = \{0, k, n\} \ (2k < n).$$

Consider the two cases where n is odd and even. By dilation we may further assume that k and n are relatively prime.

(1) If n is odd

Let $S = ((-\infty, -n-1] \cup [-k, n-k-1] \cup [n, \infty), \chi_S)$,

$$\chi_S(x) = \begin{cases} \infty & (x \leq -n-1, x \geq n) \\ 1 & (0 \leq x \leq n-k-1, -k \leq x \leq -1) \end{cases}$$

By construction, for all $x \in I = (-\infty, -n-1] \cup [n, \infty)$,

$$|T_S(x)| = \infty$$

① If $0 \leq x \leq k-1$, $T_S = \{T + \{x\}, T + \{x-k\}\}$

② If $k \leq x \leq n-k-1$, $T_S(x) = \{T + \{x\}, T + \{x-k\}\}$

③ If $n-k \leq x \leq n-1$, $T_S(x) = \{T + \{x-k\}, T + \{x-n\}\}$

$\therefore |T_S(x)| = 2$

So every integer in [0, n-1] is covered exactly twice.

Suppose T_S is splittable. Then sets A and B exist such that every integer is covered by T_A and T_B. Consider A_0 and B_0 where

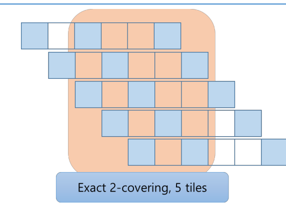
$$A_0 = A \cap [-k, n-1], B_0 = B \cap [-k, n-1].$$

Note $|A_0| + |B_0| = n$ by construction. By the pigeonhole principle, without loss of generality,

$$|B_0| \geq \tfrac{n+1}{2}. \ |A_0| = n - |B_0| \leq \tfrac{n-1}{2}.$$

Hence there are at most $\frac{n-1}{2}$ tiles in T_{A_0}, and there are two cells for each tile in [0, n-1], so the total number of cells in [0, n-1] that T_{A_0} covers is at most $n - 1$.

However there are n integers that have to be covered in the interval [0, n-1]. This is a contradiction, which implies the given covering is an unsplittable 2-covering.



<Figure 5>Example for subcase (1). A covering for $T = \{0, 2, 5\}$.

(2) If k is odd and n is even

Let $S = ((-\infty, -n-1] \cup [n+1, \infty) \cup [k-n, 0] \cup \{n-k\}, \chi_S)$, where

$$\chi_S(x) = \begin{cases} \infty & (x \leq -n-1, x \geq n+1) \\ 1 & (x = k-n, -k, 0, n-k) \\ 2 & (-k+1 \leq x \leq -1, k-n+1 \leq x \leq -k-1) \end{cases}$$

$\forall x \in \mathbb{Z} \backslash [0, n], |T_S(x)| = \infty$ by construction.

In the same idea as (1), $\forall x \in [0, n], |T_S(x)| = 2$.

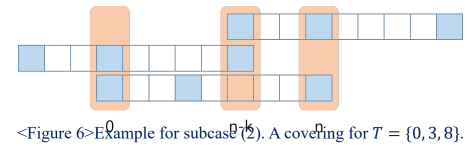
Suppose T_S is splittable. Then sets A and B exists such that every integer is covered by T_A and T_B. Consider the three integers

$$0, n-k, n.$$

Each are covered exactly twice, by three translations as follows

$$T + \{-k\}, T + \{0\}, T + \{n-k\}.$$

By the pigeonhole principle, similar to (1), WLOG, T_A cannot cover all three integers among those three, a contradiction. This implies the covering T_S is not splittable.



<Figure 6>Example for subcase (2). A covering for $T = \{0, 3, 8\}$.

Conclusion

Our main result is a characterization of the splitting number of 3-element tiles. That is, the splitting number of a 3-element tile is three if and only if it is not an arithmetic progression. The proof of our main result followed from our general construction of an unsplittable 2-covering for non-arithmetic progression tiles and our improved upper bound on the splitting number. We aim to extend these results to larger sized tiles. To this end, adapting our general construction to larger tiles and determining the tightness of our bounds is imperative. The collaboration of our group attributed to the success of our project. Along with our findings we have also found open questions and conjectures which engender further work in this area.

References

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