# A Generalization of the Dinitz Conjecture to Sudoku Boards

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**Introduction**: The first undergraduate research project I mentored was a Junior mathematics major, Eric Miller. Eric was the recipient of a Magellan Scholar Award through the University of South Carolina which provided him with a salary to conduct faculty-mentored independent research. For our project we sought to generalize the Dinitz Conjecture to Sudoku puzzles. The Dinitz Conjecture was posed in 1979 and was proved by Fred Galvin in 1994; however, the Dinitz Conjecture retains its title and can be stated as follows.

**Conjecture 1.** *(Dinitz Conjecture) The list chromatic number (or choosability) of an $n \times n$ Latin square is $n$. In other words, consider an $n \times n$ array and fix a set of $m \geq n$ symbols. For each cell pick an element of $\binom{[m]}{n}$ where the same set can be chosen for multiple cells. Is it always possible to pick an entry for each cell from its designated list so that the symbols in a row/column are distinct?*

The Dinitz Conjecture can be stated as a list edge-coloring problem for the complete bipartite graph $K_{n,n}$. Galvin resolved the conjecture by showing that the list edge-coloring number of a bipartite graph is equal to its edge-coloring number. However, Galvin's approach does not apply to the graph of a Sudoku board as the analogous Sudoku graph is not bipartite (the graph is a blow-up of $K_{n,n}$ by $K_n$).

For Miller's project we investigated the choosability of a Sudoku puzzle. Let $\mathcal{O}_n$ denote the graph of the order-$n$ Sudoku board which consists of $n^2 \times n^2$ vertices (i.e., the cells) and where two vertices are adjacent if and only if the corresponding cells are in the same row, column, or block of the order-$n$ Sudoku board. As an example, $\mathcal{O}_2 = ([16], E)$ is the graph of the $4 \times 4$ Sudoku board with four $2 \times 2$ blocks (see Figure 1 for an orientation of $\mathcal{O}_\in$).

**Conjecture 2.** *(Generalized Dinitz Conjecture) The list chromatic number of the order-$n$ Sudoku board is $n^2$.*

Denote the list chromatic number of a graph $G$ by $\mathrm{ch}(G)$. We were able to show

$$n^2 \leq \mathrm{ch}(\mathcal{O}_n) \leq 2n^2 - 2n + 1$$

so that for the simplest case $4 \leq \mathrm{ch}(\mathcal{O}_2) \leq 5$. While Galvin's proof is not applicable to a Sudoku graph, we were able to model his approach by combining graph theoretical and computational tools. Graph theoretically, one can show that the list-chromatic number of a graph is equal its chromatic number if the graph is *kernel-perfect*. A digraph is kernel-perfect if every induced subgraph has a *kernel* (i.e., an independent set where each vertex not in the set is incident to a vertex in the independent set). Richardson's theorem states that a digraph is kernel-perfect if it has no odd cycles. Thus, our approach was to construct an orientation of the edges of $O_2$ which was kernel-perfect. For induced bipartite subgraphs the result was immediate (from Galvin's proof a la Richardson's theorem). Indeed we could show $\mathrm{ch}(\mathcal{O}_2) = 4$ if we could verify

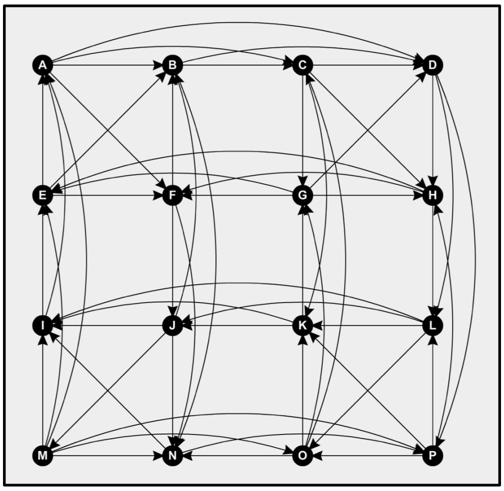


Figure 1: An orientation of $\mathcal{O}_2$.

that the induced subgraphs which had odd-cycles were kernel-perfect. Miller wrote a program in Sage to check if a particular subgraph was kernel-perfect. We further used the symmetry of the graph to reduce the search space of the computation. Unfortunately, the project ran out of time before we were able to determine the choosability of the order-2 Sudoku puzzle.

**Future Work:** The generalization of the Dinitz Conjecture to Sudoku puzzles remains open. This question is suitable for undergraduate student research because of the nature of its approach. Consider first the question of determining $\text{ch}(\mathcal{O}_2)$. To prove $\text{ch}(\mathcal{O}_2) = 4$ one would have to give an orientation of the edges of $\mathcal{O}_2$ which is kernel-perfect. I believe Miller's construction, given in Figure 1 is a contender for such an orientation. Once a particular construction has been considered, one would have to verify that all non-bipartite induced subgraphs are kernel perfect. While the number of such subgraphs is large the number of unique subgraphs up to isomorphism is considerably smaller. On the other hand, one could prove $\text{ch}(\mathcal{O}_2) = 5$ by constructing an unsolvable order-4 Sudoku puzzle where each cell could be chosen from one of any four symbols. One could even make constructions randomly or by combining uncompleted Sudoku puzzles which are unsolvable in the traditional sense. In either case, the requisite knowledge (both in graph theory and computing) can be explained to eager students with any level of background.

What makes determining $\mathcal{O}_2$ valuable for undergraduate research is that a solution (whether it be to support or deny the generalized conjecture) is insightful. List coloring questions are notoriously difficult despite their accessibility. If one could show $\text{ch}(\mathcal{O}_2) = 4$, it is likely that the generalization of the Dinitz Conjecture is true and a general argument could follow by extending the proof and providing the requisite orientation. If instead $\text{ch}(\mathcal{O}_2) = 5$ then it would hint at a strong necessary condition for kernel-perfect graphs (note that Richardson's theorem is not biconditional). In either case, this work could inspire further work at the graduate level as there are numerous open questions concerning list colorings including the choosability of claw-free graphs and Rota's basis conjecture.