The Splitting Number of an Integer Tile

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Introduction: I advised an REU style summer research program for undergraduate and high school students. This work was funded by the University of South Carolina Summer Program for Research Interns (SPRI) and Support for Minority Advancement in Research Training (SMART) program through the office of the Vice President for research.

A collection of unit disks is a *k-covering* of the plane if every point is contained in at least k disks. János Pach asked if there exists a least k such that every k-covering of the plane by unit disks has the property that the disks can be colored red and blue such that each color class covers the plane? Pach's question was answered in the negative. We consider the following discretization of Pach's problem.

Problem 1. *For a tile $T \subseteq \mathbb{Z}$ does there exist a least integer k such that every k-covering of the integers by translates of T has the property that the translates can be colored red and blue such that each color class covers the integers?*

Let $T \subseteq \mathbb{Z}$ be an integer tile and let $X \subseteq \mathbb{Z}$ be a multiset of translates. A multiset of tiles $\mathcal{T} = T + X$ is a *k-cover* of $\mathbb{Z}$ if each integer is contained in at least k elements of $\mathcal{T}$. If $\mathcal{T}$ is a 1-cover for $\mathbb{Z}$ we say that $\mathcal{T}$ is a *cover* of $\mathbb{Z}$, or equivalently that $\mathcal{T}$ covers $\mathbb{Z}$. We say $\mathcal{T}$ is *splittable* if $\mathcal{T}$ has the property that it can be partitioned into two multisets both of which are themselves coverings of the integers. For a given tile T we may ask whether there is a k so that every k-covering of $\mathbb{Z}$ with translates of T is splittable. If so, we denote the least such k by $\sigma(T)$ and refer to it as the *splitting number* of T. If no such k exists, we say that T is unsplittable. We restate Problem 2 in the following way.

Problem 2. *For a given tile T, what is $\sigma(T)$?*

We were able to answer Problem 2 in the affirmative for finite tiles. We first showed that the splitting number of a tile is at most the size of the tile, in particular,

$$2 \leq \sigma(T) \leq |T|.$$

We then considered the case when $|T| = 3$ as our cardinality bound gives $\sigma(T) \in \{2, 3\}$. Using elementary results from additive combinatorics we proved the following characterization for the splitting number of three-element tiles.

Theorem 1. *Let T be a tile where $|T| = 3$. We have $\sigma(T) = 2$ if and only if T is an arithmetic progression. Otherwise, $\sigma(T) = 3$.*

We further considered tiles which are derived from the incidence matrix of a hypergraph. Here the rows/columns of the incidence matrix are indexed by vertices/edges, respectively. We can view the incidence matrix of a hypergraph $\mathcal{H}$ as a snapshot of an $|E(\mathcal{H})|$-covering of an interval of length $|V(\mathcal{H})|$ by letting non-zero entries denote a cell from a translate and vice versa. As an example, consider the incidence matrix of the Fano plane which we use to induce a 3-covering of an interval of length 7,

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \boxtimes & \boxtimes & \square & \boxtimes & \square & \square & \square \\ \boxtimes & \square & \boxtimes & \square & \square & \boxtimes & \square \\ \boxtimes & \square & \square & \square & \boxtimes & \square & \boxtimes \\ \square & \boxtimes & \boxtimes & \square & \boxtimes & \square & \square \\ \square & \boxtimes & \square & \square & \square & \boxtimes & \boxtimes \\ \square & \square & \boxtimes & \boxtimes & \square & \square & \boxtimes \\ \square & \square & \square & \boxtimes & \boxtimes & \boxtimes & \square \end{bmatrix}$$

where $\boxtimes$ denotes a non-empty cell. In this way, we can relate the splitting of a k-covering to the coloring of a k-uniform hypergraph. We have shown that a covering is splittable if and only if the hypergraph induced by the covering over a finite interval has property B (i.e., is bipartite) for every finite interval of the integers. This is particularly useful as we can apply the probabilistic method to show

$$\sigma(T) \leq \log_2(|T - T|) + 1 + \log_2(e)$$

so that $\sigma(T) = O(\log(|T|))$.

Notice that our probabilistic bound of $\sigma(T)$ is stronger than the cardinality bound in general. In particular, $\sigma(T)$ depends on $|T-T|$. Recall Freiman's Theorem which says that if $|T-T| \leq c|T|$ then T is contained in a proper d-dimensional generalized arithmetic progression (or GAPs for short) P of size at most $k|T|$ where d and k depend only on c. Our characterization of three-element tiles demonstrates the relationship between a tile having a low splitting number and it being a (generalized) arithmetic progression. Furthermore, we can show that there exist tiles of arbitrarily large splitting number by appealing to a result of Radahakrishnan and Srinivasan wherein they showed that least number of edges of a k-uniform hypergraph which does not have property B is $\Omega(2^k\sqrt{k/\log k})$.

Future Work: There are numerous avenues for undergraduate research related to this project. One direction is to generalize the definition of splitting to *r-splitting*. That is, a k-covering is r-splittable if the covering can be partitioned into r multisets where each partition covers the integers. Furthermore, denote the r-splitting number of a tile T by $\sigma_r(T)$. In this way, a splitting (in the aforementioned sense) is a 2-splitting and $\sigma(T) = \sigma_2(T)$. It is natural to suspect that there is a relationship between k-coverings which are r-splittable and k-uniform hypergraphs which are r-colorable. Note that $\sigma_r(T)$ is non-decreasing in r, but it is likely that a stronger statement exists (for certain families of tiles, at least).

We also considered the computability of $\sigma(T)$. By our previous bounds we have that T can take on only finitely many values. It is natural to ask how difficult it is to determine $\sigma(T)$ for a given T. Note that in order to determine $\sigma(T)$ one would have to show that *every* $\sigma(T)$-covering is splittable and furthermore provide a $(\sigma(T)-1)$-covering which is not splittable. In terms of an algorithmic approach, the difficulty is in checking "every" $\sigma(T)$-covering. We first want to know if $\sigma(T)$ can be computed in a finite amount of time. The next question would be how quickly can this computation be performed in general? We considered the question briefly during our summer program and noted that this question could be answered by appealing to local properties of a covering. In particular, for a given k-covering of the integers by translates of a tile T the only translates which can cover a particular integer x must begin in the range $[x - |T|, x]$.

We have provided two upper-bounds for $\sigma(T)$. Given that the stronger bound is probabilistic, it is likely that the explicit bound given is not sharp in general. It would be interesting to know if a stronger explicit bound exists (even if that bound was identical asymptotically). Related to this question is the notion of the *spectrum* of n-element tiles,

$$\Sigma_n = \{\sigma(T) : |T| = n\}.$$

In particular, do there exist gaps (pun intended) in Σ_n or are all values between 2 and the maximum value achievable? A preliminary question to this would be to prove that for all $n \geq 2$ there exists a tile T_n such that $\sigma(T_n) = n$ which also relates to the aforementioned question about computability.

Another line of research we considered was the splitting number of infinite tiles. Preliminary results suggest that that the results for finite tiles do not extend naturally to infinite tiles. As an example, $\sigma(\mathbb{N}) = 1$ so that the lower bound for the splitting number of finite tiles no longer holds. There were two natural questions we considered in this direction. The first question is an analogue of the constructability question: given $n \geq 2$ does there exist an infinite tile T_n such that $\sigma(T_n) = n$? We further have an unlimited number of interesting questions in determining the splitting number of interesting integer sequences (e.g., the primes, Fibonacci numbers, powers of two, etc.).